OPA 1000 APR 3 1946

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1028

EXTENSION OF THE CHAPLYGIN PROOFS ON THE EXISTENCE OF COMPRESSIBLE-FLOW SOLUTIONS TO THE SUPERSONIC REGION

By Theodore Theodorsen

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

Washington
March 1946

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 1028

EXTENSION OF THE CHAPLYGIN PROOFS ON THE EXISTENCE OF COMPRESSIBLE-FLOW SOLUTIONS TO THE SUPERSONIC REGION

By Theodore Theodorsen

SUMMARY

It has been known for some time that the velocity of sound is not the upper limit for potential flow. S. A. Chaplygin in his paper "On Gas Jets" (NACA TM No. 1063) carried out some interesting proofs on the existence of solutions and gave proofs relating to maxima and minima of certain functions. In the present paper these proofs are extended to include the supersonic potential-flow field adjacent to the subsonic region treated by Chaplygin.

THEORY OF COMPRESSIBLE FLOW

The compressible-flow equations are as follows: For steady flow the law of conservation of matter gives

$$\operatorname{div} \rho\bar{q} = 0$$

For two-dimensional steady flow the relations are

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x} &= \frac{\rho_o}{\rho}\frac{\partial\psi}{\partial y} \\ \frac{\partial\varphi}{\partial y} &= -\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial x} \end{aligned}\right\} \tag{1}$$

where φ is the potential function defined by the relation

$$\varphi = \int \bar{q}\, d\bar{s}$$

and ψ is the scalar stream function which may be defined by the similar relation

$$\psi = \int \frac{\rho}{\rho_o} \bar{q} \times d\bar{s}$$

The x component of the velocity is

$$u = \frac{\partial \varphi}{\partial x} = \frac{\rho_o}{\rho} \frac{\partial \psi}{\partial y}$$

and the y component of the velocity is

$$v = \frac{\partial \varphi}{\partial y} = -\frac{\rho_o}{\rho} \frac{\partial \psi}{\partial x}$$

It is of interest to indicate a very simple relation for two-dimensional irrotational flow involving a new set of independent variables. Consider a stream tube (fig. 1) having a rectangular system of coordinates as shown with h perpendicular to and l along the streamline. If two perpendiculars to the streamline are erected at a distance dl apart, they will intersect at m. If the radius of curvature of the flow is denoted by r, from the condition of irrotationality

$$qr = \text{Constant}$$

or

$$\frac{\partial q}{\partial h} = -\frac{q}{r}$$

From the condition for conservation of matter

$$\rho q f = \text{Constant}$$

where f is the cross section of the tube; therefore,

$$\frac{\partial q}{\partial l} \frac{d\frac{1}{\rho q}}{dq} = \frac{df}{dl} \times \text{Constant}$$

or

$$\frac{\partial q}{\partial l}\left(\rho q \frac{d\frac{1}{\rho q}}{dq}\right) = \frac{1}{f}\frac{df}{dl}$$

If the angle of the stream tube with some fixed direction is θ and if r is the radius of curvature of the streamline,

$$r\,d\theta = -\,dl$$

or

$$\frac{1}{r} = -\frac{\partial\theta}{\partial l}$$

and if R is a radius of convergence of the streamlines, or the radius of curvature of the potential lines

$$\frac{1}{f}\frac{df}{dl} = \frac{1}{R} = \frac{\partial\theta}{\partial h}$$

These two equations thus may be written

$$\frac{\partial q}{\partial h} = q\frac{\partial\theta}{\partial l}$$

and

$$\frac{\partial q}{\partial\varphi}\left(q\frac{d\frac{1}{\rho q}}{dq}\right) = \frac{\partial\theta}{\partial h}$$

Since $d\varphi = q\,dl$ and $d\psi = \rho q\,dh$

$$\left.\begin{aligned}\frac{\partial q}{\partial\psi} &= \frac{q}{\rho}\frac{\partial\theta}{\partial\varphi}\\ \frac{\partial q}{\partial\varphi}\left(q\frac{d\frac{1}{\rho q}}{dq}\right) &= \frac{\partial\theta}{\partial\psi}\end{aligned}\right\} \tag{2}$$

This form of the equations of motion of a compressible fluid is given by Chaplygin (reference 1).

By interchanging the variables these equations may finally be written in the form

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\theta} &= \frac{q}{\rho}\frac{\partial\psi}{\partial q}\\ \frac{\partial\varphi}{\partial q} &= q\frac{d\frac{1}{\rho q}}{dq}\frac{\partial\psi}{\partial\theta}\end{aligned}\right\} \tag{3}$$

This form of the equations of motion was first obtained by Molenbroek (reference 2). Since q and θ are the independent variables, it is called the hodograph form of the equations of motion. All the equations are of the form

$$\frac{\partial X}{\partial x} = A\frac{\partial Y}{\partial y}$$

$$\frac{\partial X}{\partial y} = B\frac{\partial Y}{\partial x}$$

where A and B are functions of q only.

The Jacobian is

$$J = \frac{\partial(X,Y)}{\partial(x,y)} = \begin{vmatrix}\frac{\partial X}{\partial x} & \frac{\partial X}{\partial y}\\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y}\end{vmatrix}$$

$$= A\left(\frac{\partial Y}{\partial y}\right)^2 - B\left(\frac{\partial Y}{\partial x}\right)^2$$

For $J = 0$, $\frac{dy}{dx} = \sqrt{\frac{A}{B}}$. The Chaplygin proofs of uniqueness of the solution may now be extended to a region bounded in part by the line $J = 0$. The differential equation for ψ obtained from equations (3) is

$$\frac{\partial}{\partial q}\left(\frac{q}{\rho}\frac{\partial\psi}{\partial q}\right) - q\frac{d\frac{1}{\rho q}}{dq}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

If there are two values of ψ corresponding to a given fixed boundary condition, the difference between these two values may be considered a solution. The function $\psi_3 = \psi_1 - \psi_2$ is then zero along the boundary, as assumed by Chaplygin. Multiplying the left-hand side of this equation by $\psi\, dq\, d\theta$ and integrating by parts gives the following equation

$$-\iint\left[\frac{q}{\rho}\left(\frac{\partial\psi_3}{\partial q}\right)^2 - q\frac{d\frac{1}{\rho q}}{dq}\left(\frac{\partial\psi_3}{\partial\theta}\right)^2\right]dq\, d\theta$$

$$+\int\left[\frac{q}{\rho}\left(\psi_3\frac{d\psi_3}{dq}\right)d\theta - q\frac{d\frac{1}{\rho q}}{dq}\left(\psi_3\frac{d\psi_3}{d\theta}\right)dq\right] = 0$$

Since $\psi_3 = 0$ on the boundary, the second integral must be zero.

In the first integral it can be seen that the integrand does not change sign except on the line $J = 0$, which does not traverse the domain considered. The integrand, moreover, is always positive when the line $J = 0$ does not traverse the region. In such a region, therefore, $\frac{\partial\psi_3}{\partial q}$ and $\frac{\partial\psi_3}{\partial\theta}$ must be zero everywhere to avoid a contradiction. Thus, the solution is unique if the line $J = 0$ lies on the boundary or outside the region. It is important to notice that this condition does not restrict q to a value less than c as in the original proof by Chaplygin. The following important theorem has therefore been proved:

<u>The solution of the compressible potential-flow field is unique if the region is not traversed by a line $J = 0$.</u>

PROOFS FOR MAXIMA AND MINIMA OF CERTAIN FUNCTIONS

If ψ has a maximum in the field not traversed by a line $J = 0$, there may be drawn a line for which $\psi = \text{Constant} = \psi_0$ surrounding the peak value. If $\psi - \psi_0$ is considered in the same manner as in the preceding uniqueness proof, this difference value along the boundary is again zero and the line integral again becomes zero. There is left the surface integral, which can only become zero if $\psi - \psi_0$ is everywhere equal to zero.

Hence, ψ cannot have a maximum in any point of a region not traversed by a line $J = 0$.

It should be noted that this proof is different from the proof used by Chaplygin and that the result also differs in that the region of Chaplygin has been extended to the line $J = 0$, which really constitutes the limit of potential flow.

Without going into detail, it may be shown similarly that φ can have no maximum or minimum in a field not traversed by the line $J = 0$. Even more important is the fact that $\frac{\partial\psi}{\partial\theta}$ and $\frac{\partial\varphi}{\partial\theta}$ have no maxima in the field. This result follows immediately since both are solutions of the flow equation and the corresponding equation for φ. Since $\frac{\partial\psi}{\partial\theta}$ and $\frac{\partial\varphi}{\partial\theta}$ are equal to ρqR and qr, respectively, neither ρqR nor qr can have extrema in the field. It may further be shown by an identical treatment that q and ρq have no extrema in the field; it follows, therefore, that R and r show no extrema in the field and the following useful theorem is obtained:

The radius of curvature and the radius of convergence of the streamlines have their extreme values along the boundary.

From this theorem it is obvious that the points of greatest curvature (smallest radius of curvature) and the points of greatest divergence (smallest radius of convergence) are both to be found at the boundaries. An infinite radius of curvature (straight streamlines) cannot exist at isolated points in the field.

The expression for $J = 0$

$$J = \frac{\partial(X,Y)}{\partial(x,y)} = A\left(\frac{\partial Y}{\partial y}\right)^2 - B\left(\frac{\partial Y}{\partial x}\right)^2 = 0$$

may also be written

$$J = A\left(\frac{\partial Y}{\partial y}\right)^2 - \frac{1}{B}\left(\frac{\partial X}{\partial y}\right)^2 = 0$$

The problem of the change in the sign of the Jacobian will now be considered. It is first assumed that the

Jacobian may be zero along a curve in the field, and thus the existence of a negative Jacobian in a part of the field is assumed. It can be shown that such a line $J = 0$ must coincide with a particular isobaric line $q =$ Constant.

By use of a special form of the Jacobian

$$J = \frac{\partial(q, \theta)}{\partial(h, l)} = q\left(\frac{\partial\theta}{\partial c}\right)^2 - \frac{1}{\rho q \dfrac{\partial \frac{1}{\rho q}}{\partial q}} \left(\frac{\partial\theta}{\partial h}\right)^2$$

there is, for $J = 0$,

$$\frac{dl}{dh} = \sqrt{\left(\frac{q}{c}\right)^2 - 1}$$

Since $\frac{dl}{dh} = \tan \psi$, where ψ_1 is the angle which the line $J = 0$ forms with a line perpendicular to the stream-lines, it is seen that the component of the velocity q normal to the line $J = 0$ is equal to the velocity of sound c.

Using the hodograph form (3) of the equations of motion, it is seen that $x = q$, $y = \theta$, $A = q \dfrac{d\frac{1}{\rho q}}{dq}$ and $B = \frac{q}{\rho}$. Hence $J = 0$ when or if

$$\frac{d\theta}{dq} = \sqrt{-\frac{q \dfrac{d\frac{1}{\rho q}}{dq}}{q/\rho}}$$

A direct and simple integration gives for the desired curve

$$\theta - \theta_0 = f(q)$$

It can be shown in several ways that the curve $J = 0$ bordering the lower-speed region must be a straight isobaric line.

The flow lines must be parallel to each other at the points where they reach a line $J = 0$ and this line must be a straight isobar.

In three dimensions it can be seen that the surface where $q_m = c$ is a minimal surface. This minimal surface occurs because the two components $\frac{dq_y}{dy}$ and $\frac{dq_z}{dz}$ at right angles to the surface must cancel each other.

In the following remarks it is proved that a closed region cannot exist adjacent to an airfoil for which J has changed sign or equals zero:

Assume that there exists a closed region the boundary of which coincides with a curve $J = 0$. Since it has already been proved that the curve $J = 0$ in two dimensions must be a straight line, a geometrically impossible condition has been set up. A minimal surface cannot enclose any space, and in two dimensions a straight line cannot enclose any area. The Jacobian, therefore, cannot change sign in a restricted region within the flow field. Wherever $J \rightarrow 0$, a shock impends.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., September 6, 1945

REFERENCES

1. Chaplygin, S. A.: On Gas Jets. (Text in Russian.) Sci. Ann., Moscow Imperial Univ., Math.-Phys. Sec., vol. 21, 1904, pp. 1-121. (Available as NACA TM No. 1063, 1944.)

2. Molenbroek, P.: Über einige Bewegungen eines Gases mit Annahme eines Geschwindigkeitspotentials. Archiv d. Math. u. Phys. (2), vol. 9, 1890, pp.157-195.

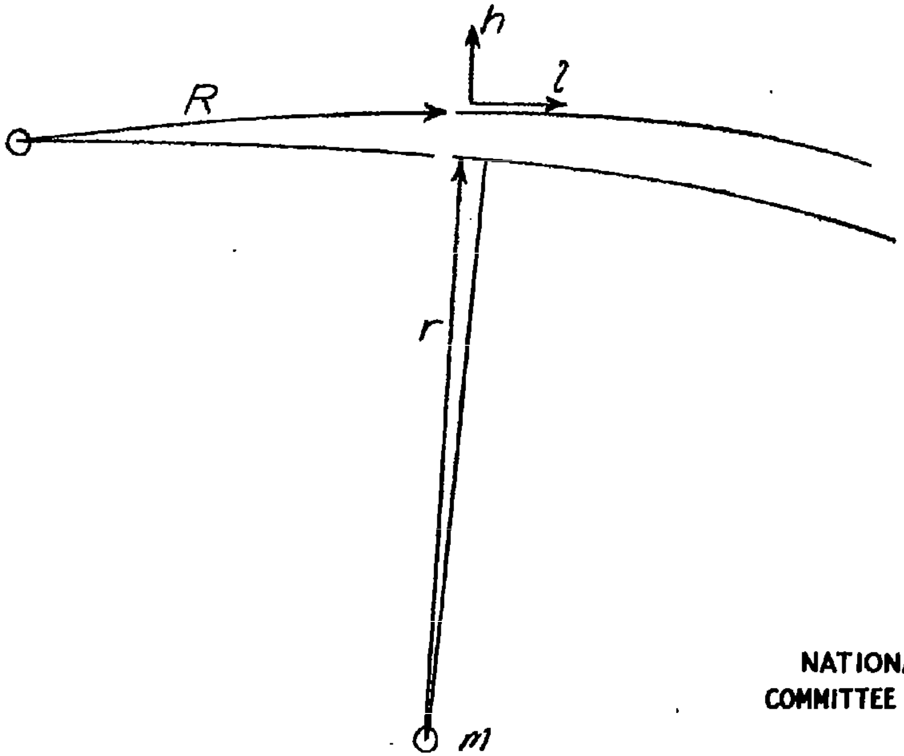

Figure 1.- Stream tube.